<u>UNIT 1</u> : INDICES

<u>Case 1</u>: Laws of Indices

The use of indices, also called **powers** or **exponents**, allows us to write products of factors and also write very large or very small numbers quickly.
For example:

$$2 \times 2 \times 2 \times 2 \text{ can be written as } 2^4$$

2^4 is read as "two to the power of four" or "two with index four"
In this case, 2 is called the **<u>base</u>** and 4 is called the **<u>index</u>**.

<u>Law 1</u>: The Multiplication Law

$$a^m \times a^n = a^{m+n}$$

<u>For examples:</u>

(a) $2^3 \times 2^4 = 2^{3+4} = 2^7$

(b) $a^5 \times a^3 = a^{5+3} = a^8$

(c) $x^2y^3 \times x^5y^4 = x^{2+5}y^{3+4} = x^7y^7$

(d) $2m^3n^7 \times 3m^2n^3 = (2 \times 3)m^{3+2}n^{7+3} = 6m^5n^{10}$

<u>Exercise 1.1</u>:

Use multiplication law of indices to simplify each of the following.
Leave your answer in index form.

(a) $3^4 \times 3^2$	(b) $3^7 \times 3^5$	(c) 4×4^5
(d) $8^6 \times 8^3$	(e) $9^8 \times 9^4$	(f) $12^5 \times 12^7$

<u>Exercise 1.2</u>: Use multiplication law of indices to simplify each of the following.

(a) $2x^3 \times x^7$	(b) $5x^6 \times x^5$	(c) $3y^4 \times y^9$
(d) $7y^7 \times y^{11}$	(e) $m^3 \times 5m^8$	(f) $p^5 \times 6p^4$

<u>Exercise 1.3</u>: Use multiplication law of indices to simplify each of the following

(a) $x^3y^4 \times x^5$	(b) $h^5k^4 \times k^5$	(c) $p^7q^6 \times p^2$
(d) $p^2q^3 \times q^4$	(e) $g^2h^5 \times h^4$	(f) $x^4y^6 \times x^7$

Law 2: The Division Law

The division law states that, when two indices are divided, their powers are subtracted provided that their bases are the same. For example:

$$a^m \div a^n = a^{m-n}$$

For examples:

(a) $2^7 \div 2^4 = 2^{7-4} = 2^3$

(b) $a^5 \div a^3 = a^{5-3} = a^2$

(c) $x^9y^6 \div x^5y^4 = x^{9-5}y^{6-4} = x^4y^2$

(d) $8m^3n^7 \div 2m^2n^3 = (8 \div 2)m^{3-2}n^{7-3} = 4mn^4$

Exercise 1.4: Use division law of indices to simplify each of the following.
Leave your answer in index form.

(a) $8^9 \div 8^3$	(b) $7^6 \div 7^2$	(c) $5^{10} \div 5^2$
(d) $15^{16} \div 15^8$	(e) $3^{15} \div 3^3$	(f) $3^{13} \div 3^7$

Exercise 1.5: Use division law of indices to simplify each of the following.

(a) $8x^4 \div 2$	(b) $9x^7 \div 3$	(c) $16x^7 \div 4x^3$
(d) $12y^8 \div 2y^2$	(e) $14x^8 \div 7x^4$	(f) $28y^9 \div 4y^3$

Exercise 1.6: Use division law of indices to simplify each of the following.

(a) $\frac{x^2y^5}{xy^2}$	(b) $\frac{6x^5y^7}{3x^2y^4}$	(c) $\frac{h^6k^{10}}{h^3k^2}$
(d) $\frac{12h^9k^7}{4h^3k^5}$	(e) $\frac{15y^8z^6}{5y^2z^3}$	(f) $\frac{20p^{10}q^8}{5p^5q^4}$

Exercise 1.7: Use laws of indices to simplify each of the following.

(a) $\frac{3x^{10}\times x^{11}}{x^{12}}$	(b) $\frac{2h^3\times 8h^7}{4h^2}$	(c) $\frac{10p^{11}\times 2p^4}{4p^6}$
(d) $\frac{8x^{12}\times 4x^7}{2x^{10}}$	(e) $\frac{32h^{12}\times 4h^2}{8h^{11}}$	(f) $\frac{5x^{10}\times 10x^8}{25x^9}$

Law 3: The Power Law

The power law states that, when an index number is raised to a certain power, the power with the index number and the outside power are multiplied. For example:

$$(a^m)^n = a^{m\times n} = a^{mn}$$

(a) $(ab)^n \Leftrightarrow a^n b^n$

(b) $(a^x b^y)^n = (a^x)^n (b^y)^n = a^{n\times x} b^{n\times y} = a^{nx} b^{ny}$

$$(a^x b^y)^n \Leftrightarrow a^{nx} b^{ny}$$

(c) $\left(\frac{a}{b}\right)^n \Leftrightarrow \frac{a^n}{b^n}$

(d) $\left(\frac{a^x}{b^y}\right)^n = \frac{(a^x)^n}{(b^y)^n} = \frac{a^{nx}}{b^{ny}}$

(e) $\left(\frac{a^x}{b^y}\right)^n \Leftrightarrow \frac{a^{nx}}{b^{ny}}$

For examples:

(a) $(2^3)^5 = 2^{3\times 5} = 2^{15}$

(b) $(2b)^3 = 2^3 b^3 = 8b^3$

(c) $(3a^2 b)^3 = 3^3 a^6 b^3 = 27a^6 b^3$

(d) $\left(\frac{a^2}{b^3}\right)^3 = \frac{a^6}{b^9}$

Exercise 1.8: Use power law of indices to simplify each of the following. Leave your answer in index form.

(a) $(5^3)^4$	(b) $(6^4)^5$	(c) $(7^5)^2$
(d) $(3^7)^6$	(e) $(a^4)^7$	(f) $(x^3)^5$

Exercise 1.9: Use laws of indices to simplify each of the following.

(a) $(a^3)^4 \times a^7$	(b) $(x^5)^2 \times x^8$	(c) $y^3 \times (y^4)^2$
(d) $h^5 \times (h^3)^6$	(e) $x \times (x^4)^5$	(f) $(y^4)^2 \times y^3$

Exercise 1.10: Use law of indices to simplify each of the following.

(a) $(3x)^3$	(b) $(2k)^5$	(c) $(x^2y)^3$
(d) $(5x^3)^2$	(e) $(x^5y^4)^4$	(f) $(5x^4y^5)^2$
(g) $(3hk^2)^3$	(h) $(4h^5k^6)^2$	(i) $(2x^3y^4)^3$

Expressing as a power of a single number

For examples:

(a) $2^3 \div 5^3 = \left(\frac{2}{5}\right)^3$

(b) $3^4 \times 7^4 = (3 \times 7)^4 = 21^4$

(c) $2^5 \div 10^5 = \left(\frac{2}{10}\right)^5 = \left(\frac{1}{5}\right)^5$

Exercise 1.11: Express each of the following as a power of a single number

(a) $2^5 \div 3^5$	(b) $7^4 \div 3^4$	(c) $3^7 \times 4^7$
(d) $2^9 \times 5^9$	(e) $14^5 \div 7^5$	(f) $18^7 \div 6^7$
(g) $3^9 \times 5^9$	(h) $4^8 \times 7^8$	(i) $24^6 \div 6^6$

Exercise 1.12: Simplify each of the following

(a) $(8x^4)^2 \times 2x^5$	(b) $(2x^4)^3 \div 4x^5$	(c) $(2xy^3)^2 \times 5xy$
(d) $81x^{10} \div (3x^2)^3$	(e) $(7a^2)^2 \times (2a^3)^3$	(f) $(2x^4)^4 \times 2$
(g) $(2a^2b)^3 \div 4a^3b$	(h) $(32ab^3)^2 \div 64ab^5$	(i) $(12xy^3)^4 \div (4xy^2)^3$
(j) $(5a^2b^4)^3 \div 25a^3b^7$	(k) $(a^2b^3)^5 \times (3ab^2)^3$	(l) $(6a^3b^2)^3 \div 36a^4b^5$
(m) $3p^2q^5 \times (2p^4q^3)^3$	(n) $8x^8y^9 \div (2xy^2)^2$	(o) $(2x^3y)^5 \div 8xy^4$

<u>Case 2:</u> Zero and Negative Indices

<u>Zero Indices</u>

When any base, which is non-zero, is raised to the power of zero, the answer is 1. That is,

$$a^0 = 1, \text{ provided that } a \neq 0$$

For examples:

(a) $3^0 = 1$, (b) $5^0 = 1$, (c) $7^0 = 1$, (d) $x^0 = 1$, (e) $y^0 = 1$, (f) $s^0 = 1$

Some further examples:

(a) $4x^0 = 4 \times x^0 = 4 \times 1 = 4$

(b) $3x^0 = 3 \times x^0 = 3 \times 1 = 3$

(c) $-5y^0 = -5 \times y^0 = -5 \times 1 = -5$

(d) $-3k^0 = -3 \times k^0 = -3 \times 1 = -3$

(e) $(4a)^0 = 4^0a^0 = 1$

(f) $(72xy)^0 = 72^0x^0y^0 = 1$

<u>Exercise 1.13</u>: Evaluate

(a) 5^0	(b) 8^0	(c) $3^0 + 4^0$
(d) 4×9^0	(e) -7×2^0	(f) $4^0 - 6^0$
(g) $(4 \times 2)^0 + 2^2$	(h) $(4b^3)^0$	(i) $2 \times 3^0 - 4$
(j) $4 - (8 \div 2)^0$	(k) $(-2ab^2)^0$	(l) $(5a^2)^0 \times (-3)^2$

Negative Indices

When attempting questions on indices, some answers will lead to a **negative power**. The answers need to be **expressed as positive powers only**. In order to do so, the following formulae are used:

$$a^{-n} = \frac{1}{a^n}$$

$$\frac{1}{a^{-n}} = a^n$$

Notes: In the above formulae:

(i) $a \neq 0$,

(ii) a^{-n} is called the reciprocal of a^n.

For examples:

(a) $a^{-3} = \frac{1}{a^3}$, (b) $3^{-1} = \frac{1}{3^1} = \frac{1}{3}$ (c) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

(d) $\frac{1}{y^{-5}} = y^5$ (e) $\frac{2}{a^{-8}} = 2a^8$ (f) $\frac{3}{x^{-4}y^{-5}} = 3x^4y^5$

Some further examples:

(a) $3x^{-2} = \frac{3}{x^2}$

(b) $4^{-2}a^{-3} = \frac{1}{4^2a^3} = \frac{1}{16a^3}$

(c) $a^{-5} \times a^3 = a^{-5+3} = a^{-2} = \frac{1}{a^2}$

(d) $8r^{-2} \div 2r^{-3} = \frac{8r^{-2}}{2r^{-3}} = 4r^{-2-(-3)} = 4r^{-2+3} = 4r$

(e) $3^{-2} = \frac{1}{3^2} = 9$

(f) $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2} = \frac{25}{9} = 2\frac{7}{9}$

(g) $\left(1\frac{1}{4}\right)^{-1} = \left(\frac{5}{4}\right)^{-1} = \left(\frac{4}{5}\right)^1 = \frac{4}{5}$

(h) $\left(2\frac{3}{4}\right)^{-2} = \left(\frac{11}{4}\right)^{-2} = \left(\frac{4}{11}\right)^2 = \frac{4^2}{11^2} = \frac{16}{121}$

Exercise 1.14: Express in terms of positive power(s) only

(a) a^{-2}	(b) b^{-4}	(c) x^{-5}
(d) $\frac{2}{y^{-2}}$	(e) $3^{-1}w^{-2}$	(f) $\frac{4}{a^{-3}b^{-4}}$

Exercise 1.15: Evaluate

(a) 4^{-2}	(b) 3^{-4}	(c) $\left(\frac{4}{3}\right)^{-2}$
(d) $\left(\frac{2}{3}\right)^{-3}$	(e)) $\left(1\frac{1}{3}\right)^{-1}$	(f) $\left(2\frac{1}{5}\right)^{-2}$

Exercise 1.16: Simplify leaving your answer as a positive index

(a) $a^7 \times a^{-12}$	(b) $b^4 \times b^{-7}$	(c) $x^9 \div x^{15}$
(d) $9^8 \div 9^{-4}$	(e) $6^0 \div 6^5$	(f) $7^{-4} \times 7^{-5}$

Further Examples including the Laws of Indices

(a) $\frac{a^{-2} \times b^{-3}}{a^{-3}b^{-5}} = a^{-2-(-3)}b^{-3-(-5)} = a^{-2+3}b^{-3+5} = ab^2$

(b) $\frac{a^{-x} \times b^{-y}}{a^{x}b^{y}} = a^{-x-x}b^{-y-y} = a^{-2x}b^{-2y} = \frac{1}{a^{2x}b^{2y}}$

(c) $\frac{4^{-2}a^{-3}b^{-6}}{3^{-3}a^{-5}b^{5}} = \left(\frac{4^{-2}}{3^{-3}}\right)a^{-3-(-5)}b^{-6-5} = \left(\frac{3^3}{4^2}\right)a^2b^{-11} = \frac{27}{16}a^2b^{-11} = \frac{27a^2}{16b^{11}}$

Exercise 1.17: Simplify each of the following

(a) $\frac{x^{-4}\times y^{-2}}{x^{-5}y^{-3}}$	(b) $\frac{x^{-3}\times x^{-2}}{x^{-5}\times x^{-1}}$	(c) $\frac{m^{-4}\times n^{-2}}{m^{2}\times n^{3}}$
(d) $\frac{2^{-3}x^{-2}y^{5}}{3\,x^{-4}y^{-6}}$	(e) $\frac{4x^{6}b^{3}}{3^{-2}x^{4}b^{3}}$	(f) $\frac{10^{-2}x^{2}b^{3}}{3^{-1}x^{-5}b^{2}}$

Case 3: Fractional Indices

A **fractional index number** consists of a base and a fractional power. Some examples are:

$$4^{\frac{1}{2}}, 8^{\frac{1}{3}}, 16^{\frac{3}{2}}, 125^{\frac{1}{3}}, \ldots$$

In general,

(i) $a^{\frac{1}{n}}$ is called the nth root of a and $a^{\frac{1}{n}} = \sqrt[n]{a}$

For examples:

$$a^{\frac{1}{2}} = \sqrt{a} \text{ (also called as the square root of } a\text{)}$$
$$a^{\frac{1}{3}} = \sqrt[3]{a} \text{ (also called as the cube root of } a\text{)}$$
$$a^{\frac{1}{4}} = \sqrt[4]{a} \text{ (also called as the fourth root of } a\text{)}$$
$$a^{\frac{1}{5}} = \sqrt[5]{a} \text{ (also called as the fifth root of } a\text{)}$$

(ii) $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m$ is called the nth root of a raised to the power of m.

For examples:

$$a^{\frac{3}{2}} = (a^3)^{\frac{1}{2}} = \sqrt{a^3}$$
$$a^{\frac{3}{4}} = (a^3)^{\frac{1}{4}} = \sqrt[4]{a^3}$$
$$a^{\frac{2}{5}} = (a^2)^{\frac{1}{5}} = \sqrt[5]{a^2}$$

Some further examples:

(i) $9^{\frac{1}{2}} = (3^2)^{\frac{1}{2}} = 3^{2\times\frac{1}{2}} = 3^1 = 3$ or $9^{\frac{1}{2}} = \sqrt{9} = 3$

(ii) $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = 2^1 = 2$ or $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

(iii) $16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2^{4\times\frac{1}{4}} = 2^1 = 2$ or $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$

Exercise 1.18: Evaluate

(a) $8^{\frac{1}{3}}$	(b) $9^{\frac{1}{2}}$	(c) $64^{\frac{1}{2}}$
(d) $125^{\frac{1}{3}}$	(e) $27^{\frac{2}{3}}$	(f) $49^{\frac{3}{2}}$

Fractional Indices with Negative Powers

For examples:

(a) $(2^3)^{-\frac{2}{3}} = 2^{3\times-\frac{2}{3}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$.

(b) $(4^{-2})^{\frac{3}{2}} = 4^{-2\times\frac{3}{2}} = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}$.

(c) $\left(1\frac{11}{25}\right)^{-\frac{1}{2}} = \left(\frac{36}{25}\right)^{-\frac{1}{2}} = \left(\frac{25}{36}\right)^{\frac{1}{2}} = \frac{25^{\frac{1}{2}}}{36^{\frac{1}{2}}} = \frac{5}{6}$

Exercise 1.19: Evaluate

(a) $(3^3)^{-\frac{2}{3}}$	(b) $(2^5)^{-\frac{2}{5}}$	(c) $(7^2)^{-\frac{1}{2}}$
(d) $(8^3)^{-\frac{1}{3}}$	(e) $(4^5)^{-\frac{3}{5}}$	(f) $(5^{-6})^{\frac{1}{2}}$
(g) $(6^{-3})^{\frac{2}{3}}$	(h) $\left(1\frac{7}{9}\right)^{-\frac{1}{2}}$	(i) $\left(2\frac{10}{27}\right)^{-\frac{1}{3}}$

Case 4: Equations involving Indices

Subcase 1: The Base in unknown

(i) $a^2 = 4$	(ii) $x^3 = 8$	(iii) $y^4 = 81$
Solution:	*Solution*:	*Solution*:
$a = \pm\sqrt{4}$	$x = \sqrt[3]{8}$	$y = \pm\sqrt[4]{81}$
$a = \pm 2$	$x = 2$	$y = \pm 3$
(iv) $a^{\frac{1}{2}} = 3$	(v) $b^{\frac{1}{3}} = 4$	(vi) $x^{\frac{2}{3}} = 4$
Solution:	*Solution*:	*Solution*:
$a = 3^2$	$b = 4^3$	$x = 4^{\frac{3}{2}} = (2^2)^{\frac{3}{2}} = 2^3$
$a = 9$	$b = 64$	$x = 8$

Exercise 1.20: Solve the following equations

(a) $x^2 = 1$	(b) $y^2 = 9$	(c) $m^3 = 27$
(d) $n^4 = 16$	(e) $a^{\frac{1}{3}} = 3$	(f) $b^{\frac{1}{4}} = 2$
(g) $x^{\frac{2}{3}} = 9$	(h) $y^{\frac{2}{5}} = 4$	(i) $a^{\frac{3}{4}} = 8$

Subcase 2: The Power in unknown

(i) $2^x = 8$	(ii) $3^x = 81$	(iii) $4^x = 64$
Solution:	*Solution*:	*Solution*:
$2^x = 2^3$	$3^x = 3^4$	$4^x = 4^3$
$x = 3$	$x = 4$	$x = 3$
(iv) $5^m = 625$	(v) $2^{x+2} = 8$	(vi) $3^{3x-1} = 81$
Solution:	*Solution*:	*Solution*:
$5^m = 5^4$	$2^{x+2} = 2^3$	$3^{3x-1} = 3^4$
$m = 4$	$x + 2 = 3$	$3x - 1 = 4$
	$x = 3 - 2 = 1$	$3x = 4 + 1$
		$x = \frac{5}{3} = 1\frac{2}{3}$

Exercise 1.21: Solve the following equations

(a) $3^x = 81$	(b) $4^x = 64$	(c) $5^x = 125$
(d) $7^{x-2} = 49$	(e) $3^{x+1} = 243$	(f) $2^{x-4} = 64$

More examples:		
(a) $(3^2)^x = 3^8$	(b) $(2^4)^x = 2^{12}$	(c) $(5^3)^x = 5^{15}$
Solution:	*Solution*:	*Solution*:
$3^{2x} = 3^8$	$2^{4x} = 2^{12}$	$5^{3x} = 5^{15}$
$2x = 8$	$4x = 12$	$3x = 15$
$\therefore x = 4$	$\therefore x = 3$	$\therefore x = 5$

Exercise 1.22: Solve the following equations

(a) $(2^3)^x = 2^9$	(b) $(2^2)^x = 2^6$	(c) $(3^2)^x = 3^6$
(d) $(4^3)^x = 4^3$	(e) $(7^2)^x = 7^4$	(f) $(9^2)^x = 9^3$
(g) $(5^3)^x = 125$	(h) $(2^3)^x = 8$	(i) $(3^3)^x = 81$

Further examples:

(a) $3^x \times 9 = 27$	(b) $3^4 \div 9 = 3^x$	(c) $(2^5)^{2x} = 2^8 \times 2^{12}$	(d) $5^{3x} = 125$
Solution:	*Solution*:	*Solution*:	*Solution*:
$3^x = \frac{27}{9} = 3$	$3^x = 81 \div 9$	$2^{10x} = 2^{8+12}$	$5^{3x} = 5^3$
$3^x = 3^1$	$3^x = 9$	$2^{10x} = 2^{20}$	$3x = 3$
$\therefore x = 1$	$3^x = 3^2$	$10x = 20$	$\therefore x = 1$
	$\therefore x = 2$	$\therefore x = 2$	

Exercise 1.23: Solve

(a) $2^x \times 8 = 64$	(b) $2^5 \div 8 = 2^x$
(c) $3^7 \div 3^{2x} = 27$	(d) $9^3 \times 27 = 3^x$
(e) $(2^3)^{2x} = 2^5 \times 2^7$	(f) $(3^2)^{4x} = (3^5)^2$
(g) $(5^2)^{4x} = (5^4)^3$	(h) $(2^4)^x = 64$

UNIT 2 : BINOMIAL EXPRESSIONS

Case 1*: Like Terms*

1. Simplify where possible:

   (a) $4a + 3a \quad = 7a$

   (b) $5x - 2x \quad = 3x$

   (c) $mn - 2mn \; = -mn$

   (d) $a^2 - 4a$ = cannot be simplified further because they are not like terms.

2. Simplify, where possible, by collecting the like terms:

   (a) $-a - 1 + 3a + 4 \quad = -a + 3a - 1 + 4$

   $= 2a + 3$

   (b) $5a - b^2 + 2a - 3b^2 = 5a + 2a - b^2 - 3b^2$

   $= 7a - 4b^2$

Exercise 2.1: Simplify, **where possible**, by collecting the like terms

(a) $5 + a + 4$	(b) $6 + 3 + a$	(c) $m - 2 + 5$
(d) $x + 1 + x$	(e) $f + f - 3$	(f) $5a + a$
(g) $-y + 3 - 3y + 7$	(h) $-8m^2 + n - m^2 + 7n$	(i) $p^2 - 3q^3 + 5p^2 - 5q^3$